Physics 798C
Superconductivity
Spring 2024
Homework 3
Due Thursday 22 February, 2024

1. Cooper Pair Size Estimate

Using the Cooper wavefunction derived in class ($\psi(\vec{r}_1 - \vec{r}_2) = \sum_{\vec{k}} g_{\vec{k}}\, e^{i\vec{k}\cdot\vec{r}}\, |00\rangle$, with $\vec{r} = \vec{r}_1 - \vec{r}_2$), show that the expectation value of the Cooper pair radius squared:

$$\langle \rho^2 \rangle = \int |\psi(\vec{r}_1 - \vec{r}_2)|^2\, (\vec{r}_1 - \vec{r}_2)^2\, d(\vec{r}_1 - \vec{r}_2) \,/ \int |\psi(\vec{r}_1 - \vec{r}_2)|^2\, d(\vec{r}_1 - \vec{r}_2)$$

is given by,

$\langle \rho^2 \rangle = \frac{4}{3}\,\frac{\hbar^2 v_F^2}{W^2}$,

where $W = -2\hbar\omega_c\, e^{-2/D(E_F)V}$ is the binding energy of the Cooper pair, and v_F is the Fermi velocity. If we say that $W \sim k_B T_c$, then estimate the size of a Cooper pair for Nb (v_F = 1.38 x 10^6 m/s).

Hints: Use the complex exponential form of the Cooper wavefunction and express factors of r in terms of the gradient on k. It is also useful to note that $\nabla_{\vec{k}}\nabla_{\vec{k}'}\delta(\vec{k} - \vec{k}') = \delta(\vec{k} - \vec{k}')\nabla_{\vec{k}}\nabla_{\vec{k}'}$. Note that the k-gradients will act upon the $g_{\vec{k}}$ and $g_{\vec{k}'}$. Recall from Lecture 5 that the weighting coefficients have the form: $g_{\vec{k}} = \frac{1}{2(\frac{\hbar^2 k^2}{2m} - E_F + \hbar\omega_c e^{-2/D(E_F)V})}$, and is isotropic in k-space. After converting the sums on k to integrals on energy, it useful to make the same approximations that we used in deriving the Cooper pairing energy, namely that the density of states, and the energy, are approximately constant in the energy range over which the attractive pairing interaction is active. Finally, it is not a bad approximation to extend the upper limits of the integrals up to infinity.

2. Equivalence of Creation/Annihilation wavefunctions and Slater Determinants

We want to show that the creation/annihilation operator format for a wavefunction is entirely equivalent to the (more laborious) Slater determinant version of the wave function. Consider the two-particle Cooper pairing wavefunction in the creation/annihilation operator format:

$$|\psi_0\rangle = \sum_{k>k_F} g_{\vec{k}}\, c^+_{\vec{k}\uparrow}\, c^+_{-\vec{k}\downarrow}\, |F\rangle,$$

where $|F\rangle$ represents the filled Fermi sea. Show that this is equivalent to the form of the Cooper pair wavefunction that we derived in class, Tinkham Eq. (3.1), by summing the two 2x2 Slater determinants with the equal coefficients $g_{\vec{k}}$ and $g_{-\vec{k}}$, which are equal by time-reversal invariance.